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A GENERALIZATION OF THE CONCEPT OF TOTAL HARMONIC DISTORTION

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When a nonlinear network is fed with a sinusoidal input signal,

$x(t) = A \sin(\omega_0 t + \Theta)$, the output signal can be written

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + \phi_n + \Theta)$$

where ω_0 is the frequency of the input signal in radians per second, ϕ_n are constants and Θ is a stochastic variable. The total harmonic distortion is then

$$d = \frac{\left[\frac{1}{2}\sum_{n=2}^{\infty} a_n^2\right]^{1/2}}{\left[\frac{1}{2}\sum_{n=1}^{\infty} a_n^2\right]^{1/2}} \tag{1}$$

Squaring and rearranging will give

$$d^2 = 1 - \frac{a_1^2}{\sum_{n=1}^{\infty} a_n^2} \tag{2}$$

The normalized crosscorrelation function between the input and the output is by definition

$$\rho_{xy}(\tau) = \frac{E\{x(t+\tau)y(t)\} - E\{x(t+\tau)\}\,E\{y(t)\}}{\sigma_x \; \sigma_y} \tag{3}$$

where σ_x and σ_y are the square roots of the input and output variances and E denotes the expected value.

For $x(t) = A \sin(\omega_0 t + \Theta)$, $E\{x(t)\}$ is zero and

$$E\{x(t+\tau)y(t)\} =$$

$$= E\{A \sin(\omega_0(t+\tau)+\Theta)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + \phi_n + \Theta)\right] = \frac{1}{2} a_1 A \sin(\omega_0\tau - \Theta$$

$$\sigma_x = \frac{A}{\sqrt{2}} \text{ and } \sigma_y = \frac{1}{\sqrt{2}}\left[\sum_{n=1}^{\infty} a_n^2\right]^{1/2}$$

Then

$$\rho_{xy}(\tau) = \frac{\frac{1}{2}\, a_1\, A \sin(\omega\tau - \phi_1)}{\frac{A}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \left[\sum_{n=1}^{\infty} a_n^2\right]^{1/2}} = \frac{a_1 \sin(\omega\tau - \phi_1)}{\left[\sum_{n=1}^{\infty} a_n^2\right]^{1/2}} \tag{4}$$

Squaring gives

$$|\rho_{xy}(\tau)|^2 = \frac{a_1^2 \sin^2(\omega\tau - \phi_1)}{\sum_{n=1}^{\infty} a_n^2} \tag{5}$$

The maximal value of $|\rho(\tau)|^2_{xy}$ is then

$$|\rho(\tau)|^2_{xy\ max} = \frac{a_1^2}{\sum_{n=1}^{\infty} a_n^2} \tag{6}$$

If we compare (2) with (6) we find that

$$d^2 = 1 - |\rho_{xy}(\tau)|^2_{max} \tag{7}$$

But the normalized crosscorrelation function is also defined through

$$\rho_{xy}(\tau) = \int_{-\infty}^{\infty} P(f) e^{j2\P f\tau}\, df \tag{8}$$

where P(f) is the Fourier transform of $\rho(\tau)$ and $j = \sqrt{-1}$

Here we have

$$P(f) = \frac{j}{2}\{e^{j\phi_1}\,\delta(f+f_0) - e^{-j\phi_1}\,\delta(f-f_0)\} \cdot \frac{a_1}{\left[\sum_{n=1}^{\infty} a_n^2\right]^{1/2}}$$ where

$\delta(\ )$ is the Dirac delta function.

It is obvious that in this case

$$|\rho_{xy}(\tau)|_{max} = \int_{-\infty}^{\infty} |P(f)|\, df \tag{9}$$

By combining (7) and (9) we end up with

$$d^2 = 1-\{\int_{-\infty}^{\infty} | P(f)|df\}^2 \tag{10}$$

This formulation may be used as a definition of nonlinear distortion also in those case where the input is not sinusoidal. It may also be used to define distortion for linear network; e.g. how the frequency transfer function of a network will distort an input signal. This means that (10) will give a measure of the total distortion created by a complex network on a specific input signal. This distortion measure will then include the interactions of the linear and non-linear parts in the network.

A complex test signal for distortion measurements should be chosen in such a way that its properties are as representative as possible for the class of messages that will be transmitted through the network.

This means that if the network is designed for transmitting speech, the test signal shall have the same spectral density and the same amplitude density functions as normal speech.

It should be pointed out that distortion measurements according to formula (10) is insensitive to the phase distortion of the network. When the input signal is speech and the output signal will be listened to monaurally, this could be justified as the ear is rather insensitive to phase in this condition. A certain influence of the phase distortion will though be achieved if the signals are segmented into short intervals and (10) are applied to these. That is, the estimate of the distortion in (10) is a stochastic variable, whose density function and thereby its mean we can estimate.

In other cases where the phase distortion of the network is essential, formula (7) is more suited for determining the distortion.